MATH 521A: Abstract Algebra

Exam 2 Solutions

1. Let $R = \mathbb{Z}, U = 5\mathbb{Z}$, two rings. Suppose $U \subseteq V \subseteq R$, and V is a ring. Prove that $V = U$ or $V = R$.

   Suppose $V \neq U$. Then there is some element $5q + r \in V$ with $0 < r < 5$. Since $U \subseteq V$, also $5q \in V$, so $r \in V$. We prove $S = \{1, 2, 3, 4\} \in V$ in four cases: $r = 1$: $S = \{r, r + r, r + r + r, r + r + r + r\}$; $r = 2$: $S = \{r + r + r - 5, r, 5 - r, r + r\}$; $r = 3$: $S = \{r + r - 5, 5 - r, r, r + r + r - 5\}$; $r = 4$: $S = \{5 - r, 10 - r - r, r + r - 5, r\}$. Lastly, since $5q \in V$ for all q, in fact $R \subseteq V$.

2. For ring R, and $x, y \in R$, define the *centralizer of* x, as $C_x(R) = \{a \in R : ax = xa\}$. Prove that $C_x(R)$ is a subring of R.

   Four things to check: (1) $0_R x = 0 = x 0_R$, so $0_R \in C_x(R)$. (2) If $a, b \in C_x(R)$ then $ax = xa, bx = xb$. Adding, we get $ax + bx = xa + xb$, and by distributivity (twice), we get $(a + b)x = x(a + b)$. Hence $a + b \in C_{xy}(R)$. (3) Suppose $a, b \in C_{xy}(R)$. We have $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$. Hence $ab \in C_{xy}(R)$. (4) Suppose $a \in C_{xy}(R)$. We have $(-a)x = -(ax)$ (by theorem), and $-(ax) = -(xa) = x(-a)$ (by theorem again). Hence $-a \in C_{xy}(R)$.

3. Let S be the ring of all continuous real-valued functions defined on $[0, 1]$, with the natural ring operations $(f \oplus g)(x) = f(x) + g(x)$, $(f \odot g)(x) = f(x)g(x)$. Define $\phi : S \to \mathbb{R}$ as $\phi : f \mapsto f(1/2)$. Prove that ϕ is a homomorphism, and find its kernel and image.

   We have $\phi(f \oplus g) = (f \oplus g)(1/2) = f(1/2) + g(1/2) = \phi(f) + \phi(g)$, and $\phi(f \odot g) = (f \odot g)(1/2) = f(1/2)g(1/2) = \phi(f)\phi(g)$. This proves ϕ is a homomorphism. We prove $Im\ \phi = \mathbb{R}$; let $c \in \mathbb{R}$ and define $f(x) = 2cx$. Then $\phi(f) = c$. Lastly, $Ker\ \phi$ is the set of all continuous real-valued functions f defined on $[0, 1]$, that satisfy $f(1/2) = 0$.

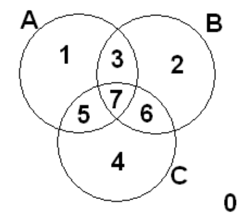
4. Prove that $\mathbb{Q}[\sqrt[3]{2}] = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} : a, b, c \in \mathbb{Q}\}$ is an commutative ring with identity.

   We have $\mathbb{Q}[\sqrt[3]{2}] \subseteq \mathbb{R}$, so we first prove it's a subring. First, $0_{\mathbb{R}} = 0 + 0\sqrt[3]{2} + 0\sqrt[3]{4} \in \mathbb{Q}[\sqrt[3]{2}]$. We have $(a + b\sqrt[3]{2} + c\sqrt[3]{4}) + (a' + b'\sqrt[3]{2} + c'\sqrt[3]{4}) = (a + a') + (b + b')\sqrt[3]{2} + (c + c')\sqrt[3]{4}$, and $(a+b\sqrt[3]{2}+c\sqrt[3]{4})(a'+b'\sqrt[3]{2}+c'\sqrt[3]{4}) = (aa'+2bc'+2cb')+(ab'+ba'+2cc')\sqrt[3]{2}+(bb'+ac'+ca')\sqrt[3]{4}$. Each are in $\mathbb{Q}[\sqrt[3]{2}]$. Lastly, $-(a + b\sqrt[3]{2} + c\sqrt[3]{4}) = (-a) + (-b)\sqrt[3]{2} + (-c)\sqrt[3]{4} \in \mathbb{Q}[\sqrt[3]{2}]$. Hence $\mathbb{Q}[\sqrt[3]{2}]$ is a ring.

   Note that $(aa' + 2bc' + 2cb') + (ab' + ba' + 2cc')\sqrt[3]{2} + (bb' + ac' + ca')\sqrt[3]{4}$ is symmetric with respect to primes, so $\mathbb{Q}[\sqrt[3]{2}]$ is commutative. We have $1_{\mathbb{Q}[\sqrt[3]{2}]} = 1 + 0\sqrt[3]{2} + 0\sqrt[3]{4}$ because $(1a' + 2 \cdot 0c' + 2 \cdot 0b') + (1b' + 0a' + 2 \cdot 0c')\sqrt[3]{2} + (0b' + 1c' + 0a')\sqrt[3]{4} = a' + b'\sqrt[3]{2} + c'\sqrt[3]{4}$. In fact, $\mathbb{Q}[\sqrt[3]{2}]$ is an integral domain.

5. Let $X = \{1,2,3,4,5\}$, and let the power set of X, denoted $\mathcal{P}(X)$, be the set of all subsets of X. Let R have ground set $\mathcal{P}(X)$, with operations $a \odot b = a \cap b$ and $a \oplus b = a\Delta b = (a \setminus b) \cup (b \setminus a) = (a \cup b) \setminus (a \cap b)$. Prove that R is a commutative ring with identity.

Associativity of $\oplus$ is annoying to check, so we use a Venn diagram. $a \oplus b$ is regions $1,5,2,6$, and so $(a \oplus b) \oplus c$ is regions $1,2,4$. On the other hand, $b \oplus c$ is regions $2,3,4,5$ and so $a \oplus (b \oplus c)$ is regions $1,2,4$.
$\oplus, \odot$ are closed since they each yield sets, so elements of $\mathcal{P}(X)$.
$a \oplus b = (a \cup b) \setminus (a \cap b) = (b \cup a) \setminus (b \cap a) = b \oplus a$. $a \odot b = a \cap b = b \cap a = b \odot a$.
We have $0_R = \emptyset$, because $a \oplus 0_R = (a \cup \emptyset) \setminus (a \cap \emptyset) = a \setminus \emptyset = a$.
We have $1_R = X$, because $1_R \odot a = X \cap a = a$.
We have $(-a) = a$, because $a \oplus a = (a \cup a) \setminus (a \cap a) = a \setminus a = \emptyset = 0_R$.
$a \odot (b \odot c) = a \odot (b \cap c) = a \cap (b \cap c) = (a \cap b) \cap c = (a \odot b) \odot c$.
Lastly, we check the annoying distributivity property, again with a Venn diagram. $b \oplus c$ is regions $2,3,4,5$, so $a \odot (b \oplus c)$ is regions $3,5$. On the other hand, $a \odot b$ is regions $3,7$ while $a \odot c$ is regions $5,7$. The symmetric difference of these two sets $\{3,7\}\Delta\{5,7\} = \{3,5\}$.



6. For ring R, $x \in R$, and $n \in \mathbb{N}$, we say x has *additive order* n if $\underbrace{x + x + \cdots + x}_{n} = 0_R$, and for $m < n$ we have $\underbrace{x + x + \cdots + x}_{m} \neq 0_R$. Define $T \subseteq R$ to be the set of those elements of R that have an additive order. Prove that T is a subring of R.

We have four things to check to apply our theorem. (1) 0_R has order 1, so $0_R \in T$. (2) Suppose $x, y \in T$, where x has order n and y has order m. We add $x + y$ nm times, and $\underbrace{(x+y) + (x+y) + \cdots + (x+y)}_{nm} = \underbrace{x + x + \cdots + x}_{nm} + \underbrace{y + y + \cdots + y}_{nm} =$

$= \underbrace{\underbrace{x + x + \cdots + x}_{n} + \cdots + \underbrace{x + x + \cdots + x}_{n}}_{m} + \underbrace{\underbrace{y + y + \cdots + y}_{m} + \cdots + \underbrace{y + y + \cdots + y}_{m}}_{n} =$

$= \underbrace{0_R + \cdots + 0_R}_{m} + \underbrace{0_R + \cdots + 0_R}_{n} = 0_R$. Hence $x + y \in T$. (3) Suppose $x, y \in T$, where x has order n and y has order m. We add xy n times, and $\underbrace{xy + xy + \cdots + xy}_{n} = (\underbrace{x + x + \cdots + x}_{n})y = 0_R y = 0_R$. Hence $xy \in T$. (4) Suppose $x \in T$, where x has order n. We have $\underbrace{(-x) + (-x) + \cdots + (-x)}_{n} = -(\underbrace{x + x + \cdots + x}_{n}) = -0_R = 0_R$. Hence $-x \in T$.